Fundamentals of

MATHEMATICS

FOR JEE MAIN AND ADVANCED

DIFFERENTIAL CALCULUS

Sanjay Mishra

Pearson

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Preface

If I am asked to choose the most important event in the history of mathematics, I shall definitely mark the simultaneous development of calculus by two contemporary, eminent mathematicians – Isaac Newton and Gottfried Leibnitz. By developing calculus, they made mathematics the only language that can describe the physical universe around us. Calculus, the mathematical analysis of motion and change, was invented by these two great mathematicians in their process of attempting to answer the fundamental questions about the world around us and the way it operates.

As we say, the Rome was not built in a day, similarly, an event so momentous involved a basic idea too, that was so profound that an average human can only hope to comprehend it. The essential idea of calculus involving the derivative and the integrals is one among such ideas, as are the paradoxes of Zeno (500 BC) and the novel idea of Archimedes (c.a 200 BC).

Calculus has major share in the syllabus of IIT JEE and other competitive examinations. During my high-school days as an IIT aspirant, and later as a tutor of mathematics, I had always felt the need for a comprehensive textbook on this subject. This book has been written with the objective of providing a textbook as well as an exercise book that focuses on problem-solving. I feel this will not only fulfill the need of class XI and class XII students but will also meet the requirements of advanced-level students who are preparing for various entrance examinations such as IIT–JEE Mains/Advanced, BIT–SAT, and other state engineering entrance examinations. This book (*Fundamental of Mathematics*, Volume-VI) has been designed to give the students a deep insight into topics such as limits, continuity and differentiability methods of differentiation and application of derivatives in detail. I have observed in my teaching career that three topics—limits, continuity and differentiability and mean value theorem, are the most challenging but high scoring topics of mathematics in the competitive exams. One of the reasons why students dread these topics is because of their non-familiarity with the basic concepts and the lack of good books that spell out the fundamentals in a student-friendly manner. This book provides a well-arranged content list that will help students and teachers to access the chapters and sub-topics of their interest conveniently. Each chapter is divided into several topics and each topic rationalizes its theory with sufficient number of worked-out problems to enable students to imbibe the concepts and apply them as required. This is followed by a textual exercise of both objective and subjective problems. Each chapter is replete with solved examples of both objective- and subjective-type questions that entail students to apply the concepts learnt in the chapter, thus enabling them acquire masters over the newly assimilated ideas. The tutorial exercise given at the end contains ample multiple-choice problems with single and multiple correct options, comprehension passage, column-matching problems and numerical integer-type questions to help students hone their mathematical skills. For teachers, this text will serve as a repository of well-graded problems, arranged topic- and subtopic-wise, that can be used to set home assignments to their students.

Suggestions for the improvement of this book are welcome and shall be gratefully acknowledged.

Sanjay Mishra

Acknowledgements

I would like to express my gratitude to Pearson Education for providing me an opportunity to share my knowledge from years of experience in teaching this comprehensive textbook. I thank all my friends and teachers for enabling me to write this book. I would like to acknowledge my pupils, without their support I would not have been able to develop new insights into the subject. I drew my inspiration to write this book in the course of my interactions with them. I feel that I have learnt more through my interaction with students than what I could have taught them. Above all I thank my parents and all my family members, who supported and encouraged me in spite of all the time it took me away from them. I am obliged to my team consisting of teachers, managers and computer operators, for their hard work and dedication in completing this task.

Sanjay Mishra

CHAPTER 1

The Limit of a Function

■ INTRODUCTION

We have studied about the important events that lead to the development of calculus, famous among these were sun, moon and earth problem, problem of force (tangent) and problem on energy (Area) that paved the path for invention of an amazing mathematical tool presently known as calculus. The breakthrough in the development of these concepts was the formulation of a beautiful mathematical idea called as limit. The story of development of this concept is too long to be told here but we would definetly mark some of the events those became the foundation stone of the concept of limits and continuity.

Zeno: Zeno was a Greek philosopher (Ca 500 BC) of an extraordinary intellect much ahead of his time primarily known for his famous paradoxes. He was mainly concerned with three problems.

(a) Problem of infinitesimals

(b) Problem of infinite

(c) Idea of continuity

Since then, the finest minds of each generation have attempted these problems. The problem of infinitesimal was solved by weierstrass whereas the solution of other two was initialed by Dedekind and concluded by 'cantor'.

***Zeno's Paradoxes*:** Zeno proposed that in a race between Achilles (a legendary Greek hero), and a Tortoise if a head start is given to the tortoise (Slower) as shown in the figure 1.1. Then it is not possible for Achilles to overtake the Tortoise. He forwarded following argument to establish his proposition.

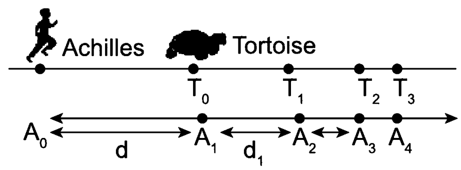

FIGURE 1.1

He said that by the time Achilles occupies the starting point of Tortoise ($A_1 = T_0$), the tortoise will have moved ahead to a new point T_1. When Achilles gets to this next position $A_2 = T_1$, the tortoise will move further ahead and occupy a new point T_2. Thus the tortoise, even though slower than Achilles, keeps moving forward. Although the distance between Achilles and the tortoise is getting smaller and smaller, the tortoise will apparently always be ahead.

By applying commonsense, one can say that Achilles must overtake the slower tortoise. But it is important to investigate that, "where is the error in Zeno's proposition"? To indicate the error in Zeno's proposition and of course to find the truth, one should sum up the infinite number of finite time intervals and prove that the summation is always finite. And this discussion shall automatically lead to the notion of limit.

Let Achilles be at point A_0 and Tortoise be at T_0 and let d be the distance between Achilles and Tortoise at the beginning of race i.e., Tortoise is given ahead start of distance d.

Corresponding	Positions of Achilles and Tortoise				
Achilles:	A_0,	A_1,	A_2,	A_3,	A_4 ….	
Tortoise:		T_0,	T_1,	T_2,	T_3,	T_4 ….

Let v_a and v_t be the speed of Achilles and Tortoise respectively. Therefore the time taken by Achilles to reach at point $T_0 = t_0 = \frac{d}{v_a}$. Now after time t_0 when Achilles reach at $T_0 = A_1$, the Tortoise would have reached at some other point T_1 at a distance d_1 from T_0 given by $d_1 = v_t \cdot t_0 = v_t \cdot \frac{d}{v_a}$. Now time taken by Achilles to go from T_0 to T_1 is given by $t_1 = \frac{d_1}{v_a} = \frac{dv_t}{v_a^2}$. In time t_1 when Achilles reach at T_1, tortoise would have reached at some other point T_2 traveling a distance $d_2 = v_t . t_1 = \frac{dv_t^2}{v_a^2}$. Now time taken by Achilles to reach at point T_2 is given by $t_2 = \frac{d_2}{v_a} = \frac{dv_t^2}{v_a^3}$. This process would go on similarly infinite number of times. Thus sum of the times taken by Achilles to reach at new position of tortoise at every stage is equal to $t_0 + t_1 + t_2 + t_3 + ...$

$$= \frac{d}{v_a}\left\{1 + \left(\frac{v_t}{v_a}\right) + \left(\frac{v_t}{v_a}\right)^2 + \left(\frac{v_t}{v_a}\right)^3 + ...\right\}$$

$= \frac{d}{v_a}\{1 + r + r^2 + r^3 + ...\}$; where $r = \frac{v_t}{v_a} < 1$ as $v_t < v_a$. Thus the above series is a decreasing infinite geometric progression and hence the sum converges to $\frac{d}{v_a}\left(\frac{1}{1-r}\right) = \frac{d}{v_a(1-r)} = \frac{d}{(v_a - v_t)}$ which is definitely a finite time interval.

Thus even if infinite number of processes of catching the tortoise by Achilles have been taken the sum of time taken by Achilles would be finite and hence definitely after a certain stage Achilles would over take the tortoise. Hence the Zeno's assumption that Achilles would never catch the tortoise when given a head start was wrong.

Similarly we can understand the meaning of "approaching a real number on real number line."

ILLUSTRATION 1: Imagine a child C_1 has a cake weighing 1 kg. He divides it in two equal parts, keeping one part with him, gives 2nd part to his friend C_2. He further divides his portion in two equal parts and gives one equal part to C_2 and go on continuously doing so. Show that at the end complete cake shall get transferred to C_2.

SOLUTION: It is very clear that C_2 gets $\frac{1}{2}$ kg cake in step 1, $\frac{1}{4}$ kg cake in step 2 and $\frac{1}{8}$ kg cake he receives from C_1 in step 3 and the process continue indefinitely. Consequently, the share of C_2 goes on increasing whereas cake held by C_1 continues decreasing and approaches zero. At the end of n steps,

Amount of cake with $C_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ upto n terms (in a GP)

$$\therefore \quad A_n = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$$

Clearly when $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$

$\Rightarrow A_n \to 1$

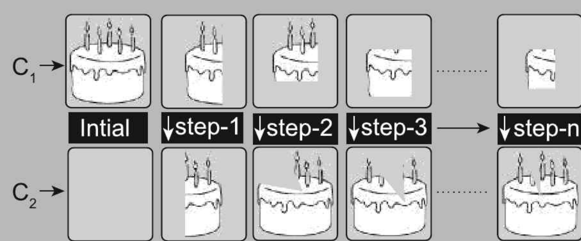

FIGURE 1.2

In mathematical language we say that as n increases and approaches to ∞, the amount of cake transformed to $C_2 = \lim_{n\to\infty}(A_n) = 1kg$

This is the basic idea and important thing is to remember, however large the value of n be the whole cake (1 kg) can never reach to C_2. This can be more clearly understood by the figure 1.2. You become more clear to see the given figure.

ILLUSTRATION 2: The sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},....$ can be described by writing a general term $\frac{n}{n+1}$ where $n = 1$, 2, 3, 4, … Can you guess the limit, L, of this sequence?

SOLUTION: The limit is an important idea in calculus, and we discuss this concept extensively later in this chapter. We will say that L is the number that the sequence with general term $\frac{n}{n+1}$ tends towards as n becomes large and larger without bound. We will define a notation to summarize this idea:

$$L = \lim_{n\to\infty}\frac{n}{n+1}$$

As you consider larger and larger values for n, you find a sequence of fractions:

$$\frac{1}{2},\frac{2}{3},\frac{3}{4},....,\frac{1{,}000}{1{,}001},\frac{1{,}001}{1{,}002}....,\frac{9{,}999{,}999}{10{,}000{,}000}..... \text{ therefore,}$$

it is reasonable to guess that the sequence of fractions is approaching the number 1.

Archimedes and The Problem of Area

The Egyptians were the first to find area of circles over as early as 3000 B.C., but Greek philosopher Archimedes first illustrated how to derive the formula for area of circle ($A = \pi r^2$) by applying an infinite limiting process, inscribing regular polygons inside circle and increasing number of sides to infinity. He called his method as "Method of exhaustion".

Considering A_n be area of n sided regular polygon inscribed in circle of radius 'r' as shown in diagrams and conclude that the square of Areas $A_3, A_4, A_5, A_6, \ldots\ldots, A_n, \ldots\ldots$ Clearly indicates that each successive area approximates more closely to that of a circle.

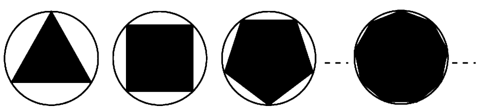

FIGURE 1.3

Based on above discussion we can define the limit of a function $f(x)$ when $x \to a$ as the real number towards which the value of function tends to approach when we approach x from left-hand side or right-hand side. So you must not confuse it with value of function at $x = a$. When x is approaching nearer and nearer to 'a' (i.e. x can be taken to as much close to 'a' as we wish), then we say that x is in neighborhood of 'a' and at that instant $f(x)$ is approaching to a real number I(say) is called limit of function. Let us study limit of a function starting from very beginning i.e. neighborhood of a point 'a'

Neighbourhood of Point '*a*'

An open interval $(a - \delta, a + \delta)$; where $\delta > 0$ is called a neighbourhood of the point 'a'. It is denoted by $N(a, \delta)$ and called as δ – *n*eighbourhood of point 'a' here 'δ' specifies the radius of neighbourhood $N(a, \delta)$, and 'a' is known as its centre for any real number $x \in N(a, \delta), \Leftrightarrow x \in (a - \delta, a + \delta)$

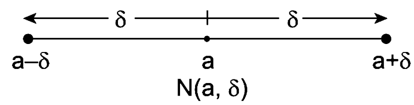

FIGURE 1.4

$\Rightarrow\ a - \delta < x < a + \delta$

$\Rightarrow\ -\delta < x - a < \delta$

$\Rightarrow\ 0 \le |x - a| < \delta$

Thus a real number x belongs to δ-neighbourhood of 'a' if and only if $0 \le |x - a| < \delta$ i.e., distance of x from 'a' is lesser than δ (may be zero) e.g., the function $f(x) = \frac{1}{\sqrt{(x-2)(3-x)}}$ has its domain $(2, 3) \equiv \left(\frac{5}{2}-\frac{1}{2},\frac{5}{2}+\frac{1}{2}\right)$ which is 1/2-neighbourhood of 5/2 i.e., neighbourhood having its centre at 5/2 and radius = 1/2.

Deleted Neighbourhood of a Point *a*

If the real number 'a' is removed from the neighbourhood $N(a,\delta)$ of 'a' then it is called a deleted neighbourhood of 'a'. Thus $(a - \delta, a) \cup (a, a + \delta)$ is called deleted neighbourhood of 'a'.

For any real ($a - \delta$, $\alpha + \delta$) number x belonging to deleted neighbourhood of 'a' we have $x \in (a - \delta, a) \cup (a, a + \delta)$

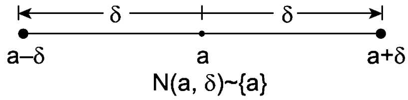

FIGURE 1.5

$\Rightarrow$ $a - \delta < x < a + \delta$ and $x \neq a$

$\Rightarrow$ $-\delta < x - a < \delta$ and $x - a \neq 0$

$\Rightarrow$ $0 < |x - a| < \delta$

Thus a real number x belongs to δ – *d*eleted neighbourhood of 'a' if and only if $0 < |x\text{-}a| < \delta$, i.e., distance of x from a is lesser than δ but not equal to zero. e.g., $f(x) = \dfrac{1}{\sqrt{(x-2)(4-x)}(x-3)}$ has its domain (2, 4) ~ {3} $\equiv (2,3) \cup (3,4)$, i.e., deleted neighbourhood of 3 having radius '1', here $a = 3$, $\delta = 1$.

Left Deleted Neighbourhood of '*a*'

The set $\{x : a - \delta, < x < a)$ is called left deleted neighbourhood of a. Thus $(a - \delta, a)$ is left deleted neighbourhood of 'a'. Thus if a real number x belongs to left deleted neighbourhood of 'a', then x is less than 'a' and distance of x from a is less than δ. e.g., the function $f(x) = \dfrac{1}{\sqrt{(x-2)(4-x)}}$ has its domain (2,4) which is left deleted neighbourhood of 4, having its radius '2'.

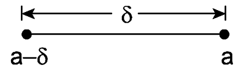

FIGURE 1.6

Right Deleted Neighbourhood of '*a*'

The set $\{x : a < x < a + \delta\}$ is called right deleted neighbourhood of a. Thus if a real number x belongs to right deleted neighbourhood of 'a', then x is greater than 'a' and its distance from 'a' is less than δ. For example the function $f(x) = \log (x - 1)(2 - x)$ has its domain (1, 2) which is right deleted neighbourhood of 1, having its radius 1.

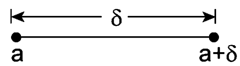

FIGURE 1.7

REMARKS:

If domain of a function f(x) is (a, b); then we can write (a, b) as

(i) $\left(\dfrac{(a+b)}{2} - \dfrac{(b-a)}{2}, \dfrac{(a+b)}{2} + \dfrac{(b-a)}{2}\right)$ *Which is the neighbourhood of* $\dfrac{a+b}{2}$ *having radius* $\dfrac{(b-a)}{2}$

(ii) *(b – (b – a), b) which is left deleted neighbourhood of b having radius (b – a)*

(iii) *(a, a + (b – a)) which is right deleted neighbourhood of a having radius (b – a). e.g., If f(x) = log (x – 2) (4 – x), then domain of f(x) =* D_f *= (2, 4) which may be defined as*

(i) Neighbourhood of 3 *having radius* 1

(ii) Left deleted neighbourhood of 4 *with radius* 2

(iii) Right deleted neighbourhood of 2 *with radius* 2

Meaning of '$x \to a$' (x tends to a)

$x \to a$ (*x tends to a*) means x is approaching nearer and nearer to 'a' but is never equal to 'a'. $x \to a$ does not predict about the way in which x is approaching to 'a' i.e., from left side of 'a' or from right side of 'a'. Thus depending on the way in which x is approaching to 'a' we define the following two symbols:

(a) $x \to a^-$: (x tends to a from negative side)

Means x is approaching to 'a' from negative side (left side). Here x is approaching to 'a' by taking the increasing values from left deleted neighbourhood of 'a' i.e., $x \in (a - \delta, a)$ and every value of x is greater than its previous value e.g., $x \to 2$ implies x takes values like 1.991, 1.992, 1.994, 1.998, 1.99901, and so on but $x < 2$ (always).

(b) $x \to a^+$: (*x* tends to *a* from positive side)

Means x is approaching to 'a' from positive side (right side). Here x is approaching to 'a' by taking the decreasing values from right deleted neighbourhood of 'a' i.e., $x \in (a, a + \delta)$ and every value of x is lesser than its previous value e.g., $x \to 2^+$ implies x takes value like 2.106, 2.102, 2.092, 2.065, 2.008 and so on but x always remains larger than 2.(x > 2).

REMARKS:

(i) $x \to a$ is equivalent to $x = a \pm h; h \to 0^+$

(ii) $x \to a^-$ is equivalent to $x = a - h; h \to 0^+$

(iii) $x \to a^+$ is equivalent to $x = a + h; h \to 0^+$

■ LIMIT OF A FUNCTION

Limit of a function at $x = a$ is tendency of the value output of the function f(x) as x gets its values nearer and nearer to a. Limit of a function can be discussed for the following two cases.

Case I : Limit of a function *f*(*x*) at a real finite number '*x* = *a*'

A real number 'ℓ' is said to be the limit of a function $f(x)$ as x tends to a if the value $f(x)$ is approaching closer & closer to ℓ as x is approaching nearer and nearer to 'a'. We can take $f(x)$, as much nearer to ℓ as we please by taking x sufficiently close to 'a'.

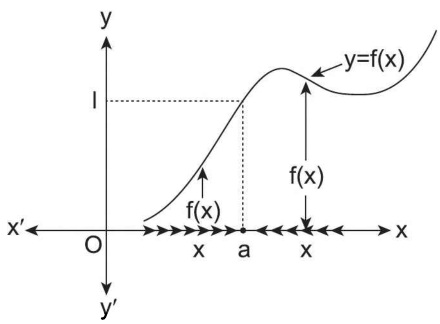

FIGURE 1.8

The above statement can be represented symbolically as $f(x) \to \ell$ as $x \to$ a and we write $\lim_{x \to a} f(x) = l$ and read as "Limit of $f(x)$ is ℓ as x tends to 'a'.

Note that $\lim_{x \to a} f(x) = l$ means $f(x)$ has the tendency to approach ℓ as x tends to 'a'. It does not ensure that $f(a) = \ell$. i.e., $f(a)$ may or may not be equal to ℓ

e.g. 1. If $f(x) = x^2$. Consider the following tables representing the values of $f(x)$ as x is approaching nearer and nearer to 2.

$x(x \to 2^-)$	1.9	1.91	1.94	1.98	1.99	1.995	1.998
$f(x)$	3.61	3.6481	3.7636	3.9204	3.9601	3.980025	3.992004

$x(x \to 2^+)$	2.05	2.04	2.02	2.01	2.005
$f(x)$	4.2025	4.1616	4.0804	4.0401	4.020025

The first one table shows that as x approaches nearer and nearer to 2 from left side, $f(x)$ is approaching nearer and nearer to 4 from left side i.e., $f(x) \to 4^-$ as $x \to 2^-$.

The second table shows that as x approaches nearer and nearer to 2 from right side $f(x)$ is approaching nearer and nearer to 4 from right side i.e., $f(x) \to 4^+$ as $x \to 2^+$.

Thus overall we conclude and say that $f(x) \to 4$ as $x \to 2$. i.e., $\lim_{x \to 2} f(x) = 4$ (Here $a = 2$, $\ell = 4$).

Graphically,

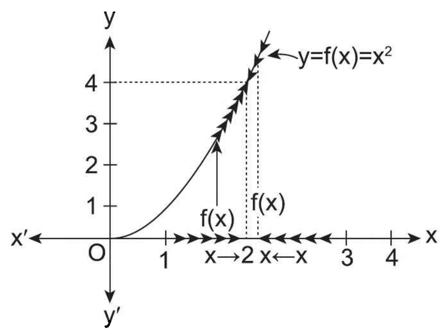

FIGURE 1.9

e.g 2. If $f(x) = \dfrac{x^2 - 9}{(x-3)} = \dfrac{(x-3)(x+3)}{(x-3)} = (x+3)$

[as $x \to 3$ implies $(x - 3)$ is non-zero and hence this factor can be cancelled out]

Consider the following tables representing the values of $f(x)$ as x is approaching nearer and nearer to 3

$x(x \to 3^-)$	2.9	2.91	2.92	2.93	2.94	2.98....
$f(x)$	5.9	5.91	5.92	5.93	5.94	5.98....

$x(x \to 3^+)$	3.05	3.04	3.03	3.02	3.01	3.004...
$f(x)$	6.05	6.04	6.03	6.02	6.01	6.004...

The first table shows that as x is approaching nearer and nearer to 3 from left side $f(x)$ is getting loser and closer to 6 from left side, i.e., $f(x) \to 6^-$ as $x \to 3^-$. The second table shows that as x is approaching nearer and nearer to 3 from right side, $f(x)$ is approaching nearer and nearer to 6 from right side i.e., $f(x) \to 6^+$ as $x \to 3^+$

Thus overall, we can say $f(x) \to 6$ as $x \to 3$. i.e., $\lim\limits_{x \to 3} f(x) = 6$

Graphically:

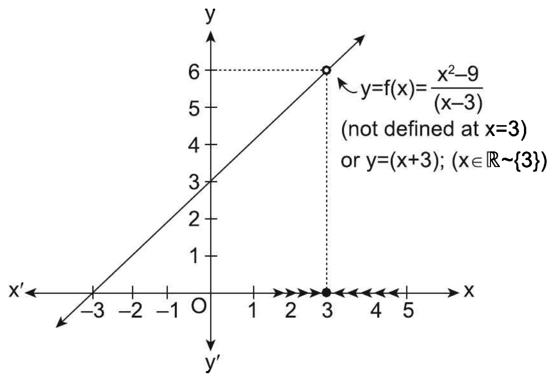

FIGURE 1.10

REMARKS:

1. *In example (1)* $\lim\limits_{x \to 2} f(x) = 4$ *and also f(2) = 4, whereas in example (2)* $\lim\limits_{x \to 3} f(x) = 6$ *but f(3) ≠ 6, thus limit of a function f(x) equal to ℓ may or may not be equal to value f(a) as x tends 'a' does not ensure f(a) = ℓ*
2. *Conversely if f(a) = ℓ, then is it necessary that* $\lim\limits_{x \to a} f(x) = l$ *?*

   The answer is no. For support consider the function f(x) = $\begin{cases} \dfrac{x^2 - 9}{x - 3} & ; x \neq 3 \\ 5 & ; x = 3 \end{cases}$ *. Here* $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$

   (As discussed earlier) and f(3) = 5. Thus f(3) = 5 but $\lim\limits_{x \to 3} f(x) \neq 5$
3. *For the functions having their graphs continuous i.e., without having any break across 'a', if* $\lim\limits_{x \to a} f(x) = \ell$*, then f(a) = ℓ and conversely if f(a) = ℓ, then* $\lim\limits_{x \to a} f(x) = \ell$.

Case II: Limit of a function *f*(*x*) at infinity (Limit at infinity)

A real number 'ℓ' is said to be the limit of a function $f(x)$ at infinity if $f(x)$ tends to ℓ as x tends to infinity ($+\infty$ or $-\infty$), i.e., $f(x)$ can be made as much close to ℓ as we please by making x sufficiently large in magnitude.

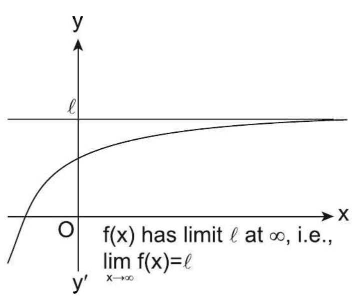

FIGURE 1.11

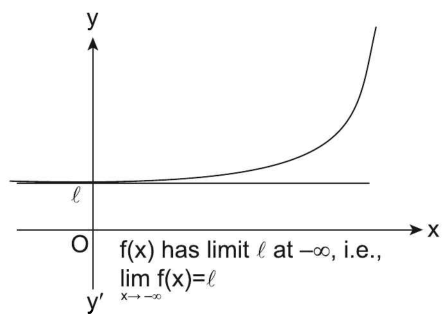

FIGURE 1.12

Illustration: Prove the following and hence draw their graph

(i) $\lim\limits_{x \to \infty} \left(1 - \dfrac{1}{x}\right) = 1$

since by increasing the values of x its reciprocal $1/x$ decrease thus

As $x \to \infty \Rightarrow \dfrac{1}{x} \to 0$

$\Rightarrow \quad 1 - \dfrac{1}{x} \to 1$. Graphically, it is shown in figure 1.13

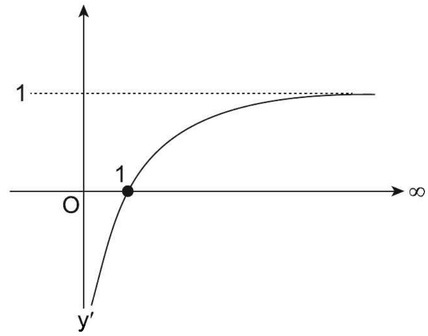

FIGURE 1.13

(ii) $\lim\limits_{x \to \infty} e^{-1/x} = 1$

when x increases from zero to infinity the reciprocal $1/x$ decreases from ∞ to zero (but takes +ve values) thus $-1/x$ increases $-\infty$ to zero taking all possible negative real values

i.e., $x \to \infty,\ \dfrac{1}{x} \to 0^+; \dfrac{-1}{x} \to 0^- \qquad \Rightarrow \quad e^{-1/x} \to 1^-$

Graphically, it is as shown below

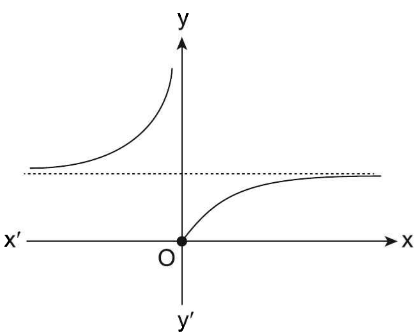

FIGURE 1.14

(iii) $\lim\limits_{x \to \pm\infty} \sin\dfrac{1}{x} = 0$

As $x \to \infty,\ \dfrac{1}{x} \to 0^+ \quad \Rightarrow \sin\dfrac{1}{x} \to 0^+$

As x takes values in $\left[-\dfrac{1}{\pi}, \dfrac{1}{\pi}\right] \sim \{0\}$ the reciprocal function $\dfrac{1}{x} \in (-\infty, -\pi] \cup [\pi, \infty)$ therefore sin $1/x$ attains its all possible values infinitely many times as the range of $1/x$ consists of infinite periodic intervals of sine function.

And As $x \to -\infty, \dfrac{1}{x} \to 0^- \Rightarrow \sin\dfrac{1}{x} \to 0^-$

Geometrically it is shown in figure 1.15

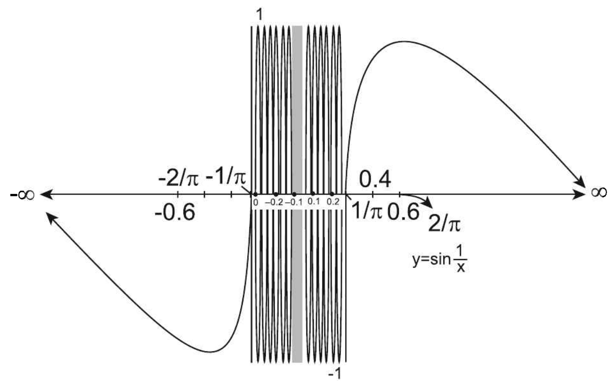

FIGURE 1.15

Why Limit of a Function is Needed?

There are some functions $f(x)$ whose values can't be determined at some real numbers (say at x = 'a'). For example

(i) $f(x) = \dfrac{\sin x}{x}$ at $x = 0$,

(ii) $f(x) = \dfrac{|x-2|}{x-2}$ at $x = 2$,

(iii) $f(x) = \dfrac{x^2-4}{x-2}$ at $x = 2$,

(iv) $f(x) = x\sin\dfrac{1}{x}$ at $x = 0$

(v) $f(x) = \dfrac{1}{x} - \dfrac{1}{\sin x}$ at $x = 0$ etc.

These functions are not defined at indicated points. However we can predict the values of real numbers (ℓ) to which these functions tend when x tends to indicated points, through the knowledge of limit i.e., $\lim\limits_{x \to a} f(x) = \ell$. We can find limit of a function at a point only when the limit is in "indeterminate form" as discussed in the next section.

■ INDETERMINATE FORMS

Some times, we come across functions which do not have definite value corresponding to some particular value of the independent variable. (If by substituting $x = a$ in any function $f(x)$, it takes up any one of form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$,

1^∞, 0^0, ∞^0, then the limit of function $f(x)$ as $x \to a$ is called indeterminate form.) There are two basic indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$ and all the other forms can be converted to these two basic forms. In such cases, value of function at $x = a$ does not exist while $\lim_{x\to a} f(x)$ may exist.

(a) $f(x) = \frac{(x^2-9)}{x-3}$. Here $\lim_{x\to 3} x^2 - 9 = 0$ and $\lim_{x\to 3} x - 3 = 0$. So $\lim_{x\to 3} f(x)$ is called an indeterminate form of type $-$.

(b) $\lim_{x\to\infty} \frac{\ln x}{x}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

(c) $\lim_{x\to 0} (1+x)^{1/x}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to 0} (\sin x)^x$ is of indeterminate form $(0)^0$

(e) $\lim_{x\to\infty} x \sin\frac{1}{x}$ is of $\infty \times 0$ form

(f) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\tan x}$ is of $\infty - \infty$ form

REMARK:

If a given limit is not of indeterminate form and the function is not defined at x = 0, we can't find it e.g., $\lim_{x\to 0} (\sin x)^{1/x}$; $\lim_{x\to 0} \left(\frac{1}{x}\right)^{1/x}$ $\lim_{x\to 0} [\ln|x|]^{1/x}$ *are not defined.*

ILLUSTRATION 3: Which of the following limits are taking up indeterminate form? Also indicate the form.

(i) $\lim_{x\to 0} \frac{1}{x}$ **(ii)** $\lim_{x\to 1} \frac{1-x}{1-x^2}$

(iii) $\lim_{x\to 0} x(\ell n\, x)$ **(iv)** $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

(v) $\lim_{x\to 0} (\sin x)^x$ **(vi)** $\lim_{x\to 0} (\ell n\, x)^x$

(vii) $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ **(viii)** $\lim_{x\to 0} (1)^{1/x}$

SOLUTION: **(i)** No **(ii)** Yes, $\frac{0}{0}$ form

(iii) Yes, $0 \times \infty$ form **(iv)** Yes, $(\infty - \infty)$ form

(v) Yes, $(0)^0$ form **(vi)** Yes, $(\infty)^0$ form

(vii) Yes, $(1)^\infty$ form **(viii)** Yes, 1^∞ form

REMARKS:

(i) *'0' doesn't mean exact zero but represents a value approaching towards zero similary to '1' and infinity.*

(ii) $\infty + \infty = \infty$

(iii) $\infty \times \infty = \infty$

(iv) $(a/\infty) = 0$ *if a is finite*

(v) $\frac{a}{0}$ *is not defined for any* $a \in R$.

(vi) $ab = 0$, *if & only if* $a = 0$ *or* $b = 0$ *and a, b are finite i.e.,* $0 \times \text{finite} = 0$

Left-hand Limit of Function

A real number 'ℓ_1' is said to be left-hand limit of a function $f(x)$, if $f(x)$ is approaching nearer and nearer to ℓ_1 if x is approaching nearer and nearer to 'a' from left side of 'a' i.e x belongs to each left deleted neighbourhood of 'a'. Symbolically we write $f(a^-) = \ell_1$ and left-hand limit is expressed as $\lim_{x \to a^-} f(x) = \ell_1$ left-hand limit is abbreviated as L.H.L. Thus L.H.L = $\lim_{x \to a^-} f(x) = l_1$.

Geometrically, it is as shown below :

(i) (Function without any break and L.H.L = ℓ_1 at $x = a$ and $f(a) = \ell_1$).

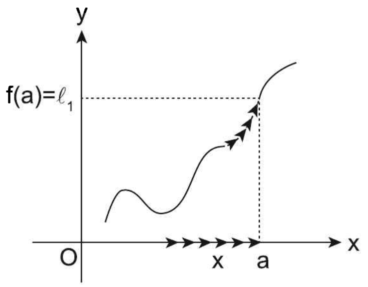

FIGURE 1.16

(ii) Function with break and L.H.L = ℓ_1 at $x = a$ and $f(a) \neq \ell_1$.

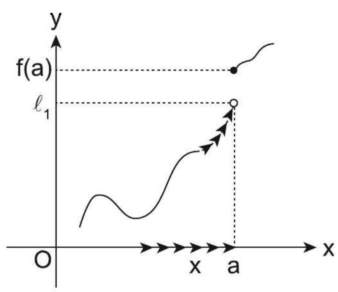

FIGURE 1.17

(iii) Function with break at $x = a$, L.H.L = ℓ_1 at $x = a$ and $f(a) = \ell_1$.

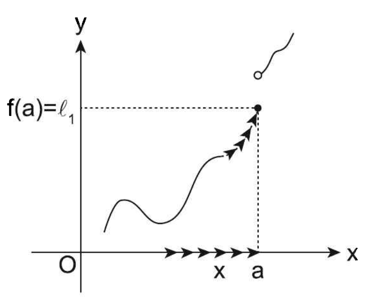

FIGURE 1.18

(iv) Function with break at $x = a$, L.H.L = ℓ_1 at $x = a$ and $f(a) \neq \ell_1$,

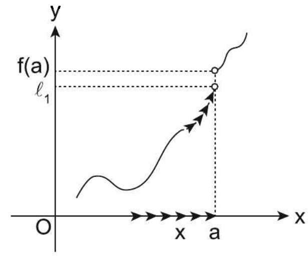

FIGURE 1.19

Example:

(i) $f(x) = (x - 1)^3$; $a = 1$, then

L.H.L = $\lim_{x \to 1^-} (x-1)^3 = 0$, as x is approaching nearer and nearer to 1 from left side $(x - 1)$ is approaching nearer and nearer to 0 from negative side i.e., $(x - 1)^3 < 0$ and $(x - 1)^3 \to 0$ i.e., $(x - 1)^3 \to 0$

(ii) $f(x) = [(x - 1)^3]$; $a = 1$; where [.] is gint. function, then

$$\text{L.H.L.} = \lim_{x \to 1^-} \left[(x-1)^3\right] = \lim_{(x-1) \to 0^-} \left[(x-1)^3\right]$$
$$= \lim_{y \to 0^-} \left[y^3\right] = -1$$

As $(y \to 0^- \Rightarrow y^3 \to 0^- \Rightarrow -1 < y^3 < 0 \Rightarrow [y^3] = -1)$

(iii) $f(x) = \dfrac{x-2}{|x-2|}; a = 2,$

then L.H.L = $\lim_{x \to 2^-} \dfrac{(x-2)}{|x-2|} = \lim_{x \to 2^-} \dfrac{(x-2)}{-(x-2)} = -1$

$$\left[\begin{array}{l} \because x \to 2^- \\ \Rightarrow x < 2 \text{ and } x \to 2 \\ \Rightarrow x - 2 < 0 \\ \Rightarrow |x-2| = -(x-2) \text{ and} \neq 0 \end{array}\right]$$

(iv) $f(x) = \sin\dfrac{1}{x}; a = 0$; then L.H.L = $\lim_{x \to 0^-} \sin\dfrac{1}{x}$ would not exist, as $x \to 0^-$ $\dfrac{1}{x} \to -\infty$ and sin θ being an oscillating function, $\sin\dfrac{1}{x}$ could not approach to a particular real number and oscillates in between –1 and 1

(v) $f(x) = \left[x \sin^2 \dfrac{1}{x}\right]; a = 0; [.]$ gint function, then L.H.L

$$= \lim_{x \to 0^-} \left[x \sin^2 \frac{1}{x}\right] =$$

[(a number approaching to 0) from left side) × (a number oscillating between 0 and 1 includingly)] = –1

Right-hand Limit of a Function

A real number 'ℓ_2' is said to be right-hand limit of a function $f(x)$ if $f(x)$ is approaching nearer and nearer to ℓ_2 as x is approaching nearer and nearer to 'a' from right side of 'a' i.e., x belongs to each right deleted neighbourhood of 'a'.

Symbolically we write $f(a^+) = \ell_2$ and it is expressed as $\lim_{x\to a^+} f(x) = \ell_2$. Right-hand limit is abbreviated as R.H.L.

Thus R.H.L $= \lim_{x\to a^+} f(x) = l_2$. Geometrically, it is as shown below

(i) Function without any break and R.H.L = ℓ_2 at $x = a$ and $f(a) = \ell_2$

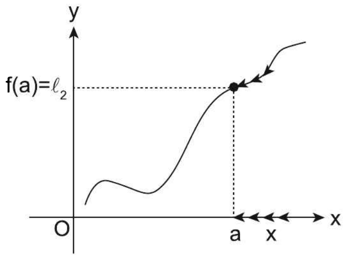

FIGURE 1.20

(ii) Function with break at $x = a$ and R.H.L = ℓ_2 at $x = a$ and $f(a) \neq \ell_2$

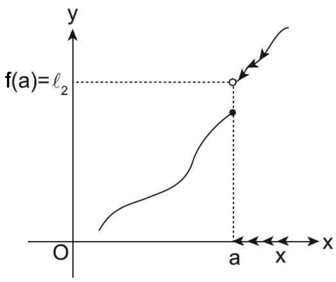

FIGURE 1.21

(iii) Function with break at $x = a$ and R.H.L = l_2 at $x = a$ and R.H.L = ℓ_2 at $x = a$ and $f(a) \neq \ell_2$

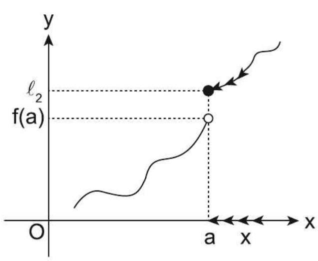

FIGURE 1.22

(iv) Function with break at $x = a$, R.H.L = ℓ_2 at $x = a$ and $f(a) \neq \ell_2$ e.g.,

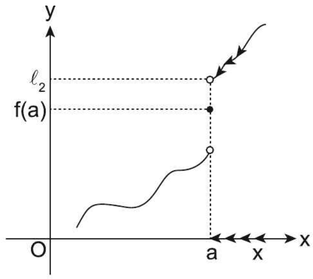

FIGURE 1.23

1. $f(x) = x^3 - 1$; $a = 1$;

   then $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(x^3 - 1\right) = 0$

   As $x \to 1^+, x^3 \to 1^+ \Rightarrow x^3 - 1 \to 0^+$

2. $f(x) = [(1 - x)^3]$; $a = 1$; where [.] is gint function

   then $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left[(1-x)^3\right] = \lim_{(x-1)\to 0^+}\left[-(x-1)^3\right]$

   $= \lim_{y\to 0^+}\left[-y^3\right] = -1 \text{ as } y \to 0^+ \Rightarrow -y^3 \to 0^-$

   $\Rightarrow -1 < -y^3 < 0 \Rightarrow [-y^3] = -1$

3. $f(x) = \dfrac{x-2}{|x-2|}; a = 2$

   Then $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\dfrac{x-2}{|x-2|} = \lim_{x\to 2^+}\dfrac{(x-2)}{(x-2)} = 1$

   (As $x \to 2^+ \Rightarrow x - 2 > 0 \Rightarrow |x - 2| = x - 2 \neq 0$)

4. $f(x) = \sin\dfrac{1}{x}; a = 0$; then $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\sin\dfrac{1}{x}\right)$

   does not exist as $x \to 0^+$

   $\Rightarrow \dfrac{1}{x} \to \infty \Rightarrow sin\dfrac{1}{x}$ oscillates in between –1 and 1)

5. $f(x) = \left[x\sin^2\dfrac{1}{x}\right]; a = 0$; [.] is gint function, then

   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[x\sin^2\dfrac{1}{x}\right] =$

   [(a number approaching 0 from right side) × (a number oscillating between 0 and 1)] $= [0^+] = 0$

Procedure to find one sided limit of a function

1. To evaluate left-hand limit of a function, we substitute $x = a - h$ and take the limit $h \to 0^+$

   i.e., $\lim_{x\to a^-} f(x) = \lim_{h\to 0^+} f(a-h)$

   e.g., for $f(x) = \dfrac{x-2}{|x-2|}$;

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{(x-2)}{|x-2|} = \lim_{h\to 0^+} \frac{(2-h-2)}{|2-h-2|}$$

$$= \lim_{h\to 0^+} \frac{(-h)}{|-h|} = \lim_{h\to 0^+} \frac{(-h)}{h} = -1$$

$(\because h \to 0^+ \Rightarrow h > 0 \Rightarrow -h < 0 \Rightarrow |-h| = h)$

2. To evaluate right-hand limit of a function, we substitute $x = a + h$ and take the limit $h \to 0^+$

i.e., $\lim_{x\to a^+} f(x) = \lim_{h\to 0^+} f(a+h)$

e.g., for $f(x) = \frac{\{x-2\}}{|2-x|}$; $\{x\}$ is fractional part of x;

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0^+} \frac{\{2+h-2\}}{|2-(2+h)|} = \lim_{h\to 0^+} \frac{\{h\}}{|-h|} = \frac{h}{h} = 1$$

$(\because 0 < h < 1 \Rightarrow \{h\} = h$ and $-h < 0 \Rightarrow |-h| = h)$

Existence of limit of a function

The limit of a function at $x = a$, is said to exist if

(i) $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$

(ii) ℓ is a finite real number

i.e., Left-hand limit and right-hand limit of function exist, equal and they are equal to finite real number. Thus existence of limit of a function at $x = a$ means "As x tends to 'a' from either way (from left or right) $f(x)$ tends to a unique finite (real number).

Geometrically it is as shown below

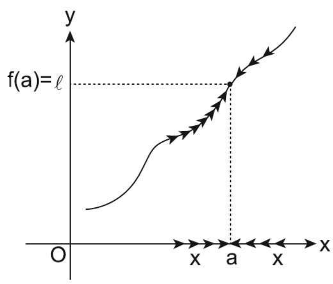

FIGURE 1.24

Function without any break at $x = a$ and L.H.L = R.H.L = ℓ at $x = a$, $f(a) = \ell$

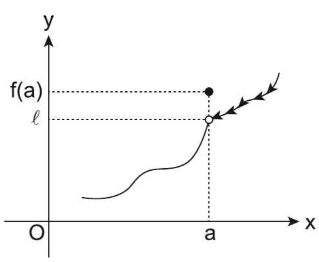

FIGURE 1.25

Function having a break at $x = a$, L.H.L = R.H.L = $\ell \neq f(a)$

Reason for non-existence of limit of a function

Any one of the following may be the reason for non-existence of limit of a function.

(i) Any one or both L.H.L and R.H.L do not exist

(ii) Both L.H.L and R.H.L exist but are unequal

(iii) $f(x)$ oscillates with large frequency near the point $x = a$

The following graphs illustrate the reasons for non-existence of limits:

(i) (L.H.L and R.H.L. exist but are not equal, x)

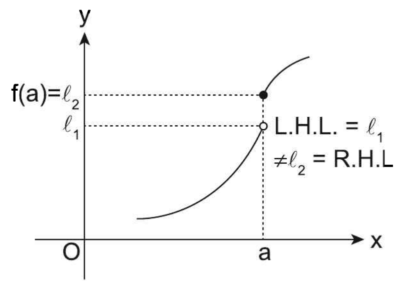

FIGURE 1.26

(ii) One of the L.H.L. and R.H.L. exist finitely and other in infinitely.

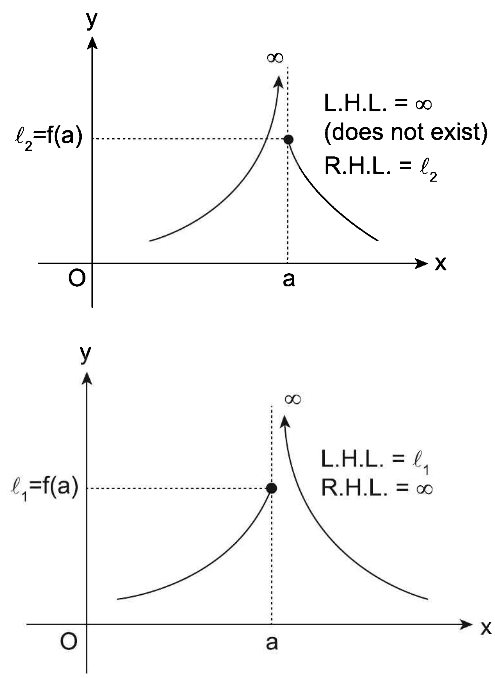

FIGURE 1.27

(iii) Both L.H.L. and R.H.L. are infinite

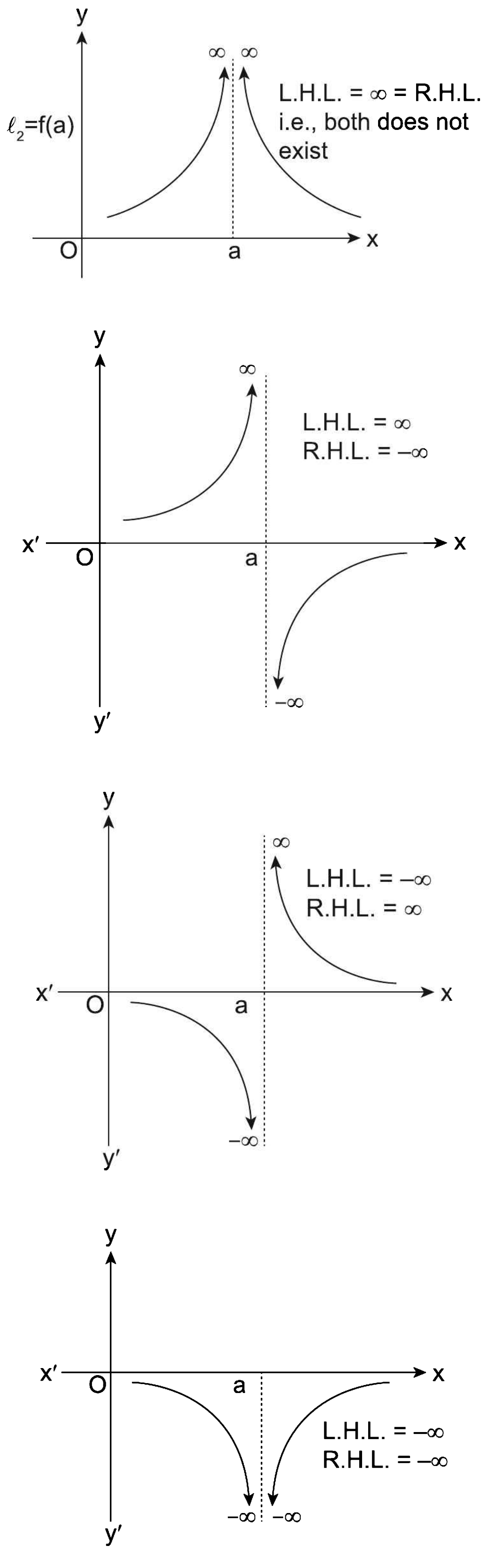

FIGURE 1.28

(iv) When the function is oscillating

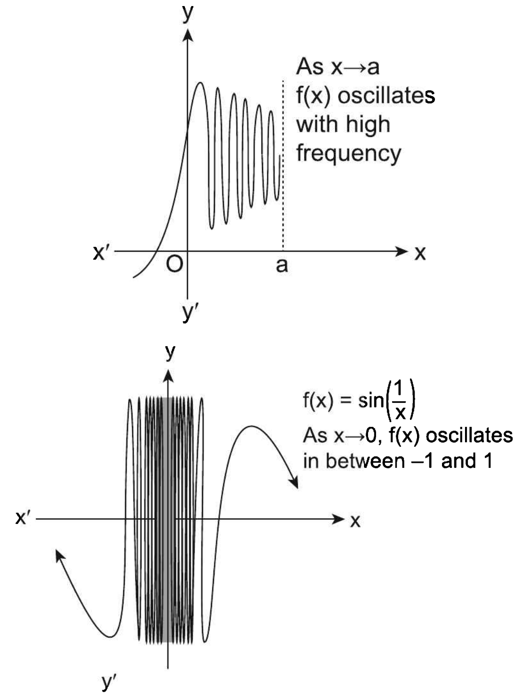

FIGURE 1.29

For example:

(i) $f(x) = \dfrac{x-3}{|x-3|}$; L.H.L $= \lim\limits_{x\to 3^-}\dfrac{x-3}{|x-3|} = -1$ and

RH.L $= \lim\limits_{x\to 3^+}\dfrac{x-3}{|x-3|} = 1$

∴ L.H.L ≠ R.H.L

∴ Limit does not exist, inspite L.H.L and R.H.L exists separately.

(ii) $f(x) = \dfrac{1}{\sin x}; a = \pi$; then

$\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-}\dfrac{1}{(\sin x)} = \infty$ and

$\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+}\dfrac{1}{(\sin x)} = -\infty$

∴ L.H.L and R.H.L does not exist

(iii) $f(x) = |\tan x|$; $a = \pi/2$, then

L.H.L $= \lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-} |\tan x| = \infty$ and

R.H.L $= \lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} |\tan x| = \infty$

∴ L.H.L and R.H.L does not exist, however both are infinite.

(iv) $f(x) = \dfrac{1}{[x-1]}$; $a = 1$, where [.] is gint function. Then

L.H.L =

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{1}{[x-1]} = \lim_{h\to 0^+}\frac{1}{[(1-h)-1]} = \lim_{h\to 0^+}\frac{1}{[-h]}$$

$$= \frac{1}{(-1)} = -1 \; (\because -1 < -h < 0)$$

Here domain of $f(x)$ is $R \sim [1, 2)$. Since $f(x)$ is not defined in [1,2), thus we need not to find R.H.L of $f(x)$ at $x = 1$ and $\lim\limits_{x\to 1} f(x)$ is considered to be $\lim\limits_{x\to 1^-} f(x)$ and is equal to – 1. Thus we in this case say limit exists and is equal to –1.

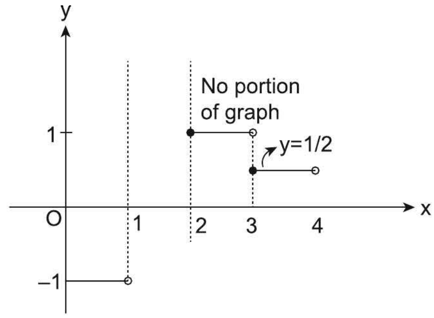

FIGURE 1.30

REMARKS:

1. *If limit of a function f(x) is to be determined at x = a first of all make sure that the function f(x) is defined in left deleted neighbourhood (a-δ, a) and right deleted neighbourhood (a, a + δ). If f(x) is defined in (a – δ, a) and is not defined in (a, a + δ), then left-hand limit is taken as the value of given limit. Similarly if f(x) is not defined in (a-δ, a) and is defined in (a, a + δ), then right-hand limit is taken as the value of given limit. For example*

   (i) $f(x) = \dfrac{1}{[x-1]}$; *([x] is gint function) is defined in* $(1 - \delta, 1); \delta > 0$ *but not defined in* $[1, 1 + \delta); 0 < \delta \le 1$

   $\therefore \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1^-} f(x) = -1$

   (ii) $f(x) = \sin^{-1}x$; $a = 1$,

   then f(x) is defined in $[1 - \delta, 1]; 0 < \delta \le 2$

   but f(x) is not defined in $(1, 1 + \delta); \delta > 0$

   Thus $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \sin^{-1} x = \dfrac{\pi}{2}$

2. *If L.H.L = R.H.L = ∞ or -∞, then we say that limit does not exist. It means the limit does not exist finitely, i.e, there is no real finite number to which f(x) tends as x tends to a. In this case we say"limits exists infinitely/"'*

3. ***Infinite Limits**: If f(x) tends to ∞ (or - ∞) as x → a (or ∞), then the limit is called infinite limit. Thus we can make f(x) as much large in magnitude as we please by making x sufficiently close to a.*

4. *By* $\lim\limits_{x\to a}$ *f(x) we mean x takes values closer and closer to 'a' without being equal a.*

5. *It is evident from the definition that in order to find the limit of f(x) at x = a, the first thing is that f(x) should be well defined in the neighbourhood of x = a and not necessarily at x = a (that means x = a may or may not be in the domain of f(x)), because we have to examine its behaviour or tendency in the neighbourhood of x = a.*

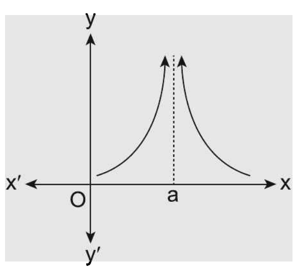

FIGURE 1.31

6. *If* $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, *then limit does not exist at* $x = a$

7. *If* R.H.L. = L.H.L = ∞, *then limit is said to exist infinitely.*

ILLUSTRATION 4: Evaluate the following one sided limits:

(i) $\lim\limits_{x\to 2^+} \dfrac{|x-2|}{(x-2)}$ **(ii)** $\lim\limits_{x\to 2^-} \dfrac{|x-2|}{(x-2)}$

(iii) $\lim\limits_{x\to 3^+} [x]$;[.] is gint function **(iv)** $\lim\limits_{x\to 3^-} [x]$,[.] is gint function

(v) $\lim\limits_{x\to -\pi/4^-} [\cot x]$. **(vi)** $\lim\limits_{x\to -\pi/4^+} [\cot x]$

(vii) $\lim\limits_{x\to -\pi/4^+} [\cot x]$ **(viii)** $\lim\limits_{x\to 10^-} [\log x]$

SOLUTION: **(i)** $\lim\limits_{x\to 2^+} \dfrac{|x-2|}{(x-2)}$; As $x \to 2^+ \Rightarrow x > 2$

$\therefore\ x - 2 > 0 \Rightarrow |x-2| = x - 2$ $\therefore\ \lim\limits_{x\to 2^+} \dfrac{|x-2|}{(x-2)} = \lim\limits_{x\to 2^+} \dfrac{(x-2)}{(x-2)} = 1$

(ii) $\lim\limits_{x\to 2^-} \dfrac{|x-2|}{x-2}$; As $x \to 2^- \Rightarrow x < 2$

$\Rightarrow x - 2 < 0$ $\Rightarrow |x-2| = -(x-2)$

$\therefore\ \lim\limits_{x\to 2^-} \dfrac{|x-2|}{(x-2)} = \lim\limits_{x\to 2^-} \dfrac{-(x-2)}{(x-2)} = -1$

(iii) $\lim\limits_{x\to 3^+} [x]$; As $x \to 3^+$

$\Rightarrow 3 < x < 4$ $\therefore\ [x] = 3$

$\therefore\ \lim\limits_{x\to 3^+} [x] = 3$

(iv) $\lim\limits_{x\to 3^-} [x]$; As $x \to 3^-$

$\Rightarrow 2 < x < 3$ $\therefore\ [x] = 2$

$\therefore\ \lim\limits_{x\to 3^-} [x] = 2$

(v) $\lim\limits_{x\to\left(-\frac{\pi}{4}\right)^-} [\cot x]$; As $x \to \left(-\dfrac{\pi}{4}\right)^-$

$\Rightarrow x < -\pi/4$

$\Rightarrow \cot x > \cot\left(-\dfrac{\pi}{4}\right)$ as cot x is a decreasing function

$0 > \cot x > -1$

$\therefore\ [\cot x] = -1$

$\therefore\ \lim\limits_{x\to\left(-\frac{\pi}{4}\right)^-} [\cot x] = -1$

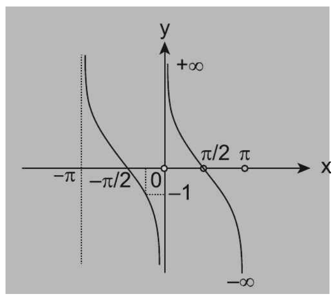

FIGURE 1.32

(vi) $\lim\limits_{x\to\left(-\frac{\pi}{4}\right)^+} [\cot x]$; As $x \to \left(-\dfrac{\pi}{4}\right)^+$ $\Rightarrow x > -\dfrac{\pi}{4}$

$\Rightarrow \cot x < \cot(-\pi/4)$

$\therefore\ [\cot x] = -2$

$\Rightarrow -2 < \cot x < -1$

$\therefore\ \lim_{x \to \left(-\frac{\pi}{4}\right)^+} [\cot x] = -2$

(vii) $\lim_{x \to 10^-} [\log x]$; As $x \to 10^-$

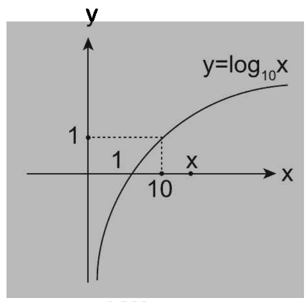

FIGURE 1.33

$\Rightarrow 9 < x < 10$

$\Rightarrow 1 < x < 10$

As $\log_{10} x$ is an increasing function

$\Rightarrow \log_{10} 1 < \log_{10} x < \log_{10} 10$

$\Rightarrow 0 < \log x < 1$

$\Rightarrow [\log x] = 0$

$\therefore\ \lim_{x \to 10^-} [\log x] = 0$

(viii) $\lim_{x \to 10^+} [\log x]$; As $x \to 10^+$

$\therefore\ x > 10$ but x is nearer to 10

$\therefore\ 10 < x < 100$

$\therefore\ \log_{10} 10 < \log_{10} x < \log_{10} 100$

$\Rightarrow 1 < \log_{10} x < 2$

$\therefore\ [\log x] = 1$

$\therefore\ \lim_{x \to 10^+} [\log x] = 1$

ILLUSTRATION 5: Evaluate the following one sided limit:

(i) $\lim_{x \to 2^-} |[x-2]|$; [.] is greatest integer function

(ii) $\lim_{x \to 3^+} \{x-2\}$; {.} is fractional part of x

(iii) $\lim_{x \to 2^-} \left[\sin \frac{(x^2-4)}{(x+2)}\right]$; [.] is greatest integer function

(iv) $\lim_{x \to 3^-} \left\{\tan\left(\frac{x^2-9}{x+3}\right)\right\}$; {.} is fractional part of x

(v) $\lim_{x \to 0^+} \left\{\cos\left(\frac{x-\pi/2}{x^2-\pi^2/4}\right)\right\}$; {.} is fractional part of x

SOLUTION: **(i)** $\lim_{x \to 2-} |[x-2]|$; As $x < 2$ and $x \to 2 \Rightarrow 1 < x < 2$

$\Rightarrow -1 < x - 2 < 0$

$\Rightarrow [x-2] = -1$

$\Rightarrow |[x-2]| = 1$

(ii) $\lim_{x \to 3^+} \{x-2\}$; As $x \to 3^+ \Rightarrow 3 < x < 4$

$\Rightarrow 1 < x - 2 < 2$

$\Rightarrow (x-2) = [x-2] + \{x-2\}$

$\Rightarrow (x-2) = 1 + \{x-2\}$

$\Rightarrow \{x-2\} = (x-2) - 1 = x - 3$

$\therefore\ \lim_{x \to 3^+} \{x-2\} = \lim_{x \to 3^+} (x-3) = 0$

(iii) $\lim_{x \to 2^-} \left[\sin\left(\frac{x^2-4}{x+2}\right)\right] = \lim_{x \to 2^-} [\sin(x-2)]$ $[\because x \to 2 \Rightarrow x + 2 \neq 0]$

$= \lim_{h \to 0^+} [\sin(2-h-2)] = \lim_{h \to 0^+} [\sin(-h)] = \lim_{h \to 0^+} [-\sin h] = [k]$; (where $-1 < k < 0$) $= -1$

(iv) $\lim_{x\to 3^-}\left\{\tan\left(\frac{x^2-9}{x+3}\right)\right\} = \lim_{x\to 3^-}\{\tan(x-3)\}$ as $x+3\neq 0$

$= \lim_{h\to 0^+}\{\tan(3-h-3)\} = \lim_{h\to 0^+}\{\tan(-h)\}$

$= \lim_{h\to 0^+}(1-\{\tan h\})$ ($\because$ $\{x\} + \{-x\} = 1$ for $x \notin \mathbb{Z}$ $= 0$ for $x \notin \mathbb{Z}$)

$= \lim_{h\to 0^+}(1-\tan h)$ ($\because$ $0 < \tan h < 1$ $\Rightarrow$ $\{\tan h\} = \tan h$)

$= 1 - 0 = 1$

(v) $\lim_{x\to 0^+}\left\{\cos\left(\frac{x-\pi/2}{x^2-\pi^2/4}\right)\right\} = \lim_{x\to 0^+}\left\{\cos\left(\frac{1}{x+\pi/2}\right)\right\}\left[\because x-\frac{\pi}{2}\neq 0\right]$

As $x \to 0^+$; $x + (\pi/2) \to \pi/2^+$

$\Rightarrow \frac{1}{x+(\pi/2)} \to \left(\frac{2}{\pi}\right)^+ \Rightarrow 0 < \frac{1}{x+(\pi/2)} < 1$ radian

$\Rightarrow 0 < \cos\left(\frac{1}{x+\pi/2}\right) < 1 \Rightarrow \left\{\cos\left(\frac{1}{x+\pi/2}\right)\right\} = \cos\left(\frac{1}{x+\pi/2}\right)$

$\therefore \lim_{x\to 0^+}\left\{\cos\left(\frac{x-\pi/2}{x^2-\pi^2/4}\right)\right\} = \lim_{x\to 0^+}\cos\left(\frac{1}{x+\pi/2}\right) = \cos\left(\frac{2}{\pi}\right)$

ILLUSTRATION 6: If $f(x) = \begin{cases} x^2+1 \ ; x \geq 1 \\ 3x-1 \ ; x < 1 \end{cases}$, then find the value of $\lim_{x\to 1} f(x)$.

SOLUTION: L.H.L $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3x-1) = 3(1) - 1 = 2$

R.H.L $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+1) = 1^2 + 1 = 2$

$\therefore$ L.H.L = R.H.L $\Rightarrow$ limit of $f(x)$ as $x \to 1$ exists and is equal to 2

ILLUSTRATION 7: Simplify $f(x) = \begin{cases} \frac{|x^2-1|}{x-1} \ ; x\neq 1 \\ 0 \ ; x = 1 \end{cases}$, and test the existence of its limit at $x = 1$ and -1.

SOLUTION: Simplifying the above function, we get $f(x) = \begin{cases} x+1 & ; \ x<-1 \\ -(x+1) & ; \ -1<x<1 \\ 0 & ; \ x=1 \\ x+1 & ; \ x>1 \end{cases}$

(i) at $x = 1$; Clearly L.H.L at $x = 1$ is -2 while R.H.L = 2, so $\lim_{x\to 1} f(x)$ does not exist.

(ii) at $x = -1$; L.H.L = R.H.L = 0 (at $x = -1$) so $\lim_{x\to -1} f(x)$ exists and it is equal, to zero.

ILLUSTRATION 8: Show that $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ does not exist.

SOLUTION: Given function is $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}$

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1} = \lim_{h\to 0}\frac{(1/e^{1/h}-1)}{(1/e^{1/h}+1)} = \frac{0-1}{0+1} = -1$